From self-consistent covariant effective field theories to their Galilean-invariant counterparts

A. Sulaksono[1,5], P. -G. Reinhard[2], T. J. Bürvenich[3], P. O. Hess[3,4] and J. A. Maruhn[5]
[1]*Departemen Fisika, FMIPA, Universitas Indonesia, Depok, 16424, Indonesia*
[2]*Institut für Theoretische Physik II, Universität Erlangen-Nürnberg, D-91058 Erlangen, Germany*
[3]*Frankfurt Institute for Advanced Studies, Universität Frankfurt, 60438 Frankfurt am Main, Germany*
[4]*Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, Apdo. Postal 70-543, México 04510 D.F*
[5]*Institut für Theoretische Physik, Universität Frankfurt, 60438 Frankfurt am Main, Germany*

We discuss how to obtain the nonrelativistic limit of a self-consistent relativistic effective field theory for dynamic problems. It is shown that the standard v/c expansions yields Galilean invariance only to first order in v/c, whereas second order is required to obtain important contributions such as the spin-orbit force. We propose a modified procedure which is a mapping rather than a strict v/c expansion.

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I. INTRODUCTION

Since the development of the special theory of relativity classical physics is viewed as its *nonrelativistic limit* for c, the velocity of light, going to infinity (or correspondingly, considering all relevant velocities to be much smaller than c). In Quantum Mechanics, Dirac's theory of the electron carrying spin 1/2 has as its nonrelativistic limit the Pauli equation.

Since then, the problem in obtaining a nonrelativistic limit of a theory has enjoyed permanent interest. Many physical systems are on the borderline between relativistic and classical physics, determined by the velocity and/or the size of the system. Prominent examples are heavy atoms and nuclei, where the spin-orbit force is a relativistic effect and shows that, although a nonrelativistic description is in general easier to handle, relativistic effects cannot always be neglected. Understanding the transition from relativistic to nonrelativistic physics while maintaining the traces of relativistic effects, is therefore of utmost interest.

One of the first, and still most prominent, attempts to handle this transition in a systematic way was the Foldy-Wouthysen transformation [1] using a canonical transformation to obtain two equations with two components, one of which becomes the Pauli equation in the nonrelativistic limit. An alternative method is related to group contractions [2], where one discusses under which conditions a group can be *contracted* to another one, involving non singular transformations. In particular, the Lorentz group $SO(3,1)$ can be contracted to the Galilei group, taking the limit $c \to \infty$ and neglecting corrections of the order of $(\frac{v}{c})^2$. On the level of group generators this contraction procedure is direct, but on that of representations it is not as trivial [2]. As we shall see, this is the case in self-consistent effective field theories involving fields with a spin $\frac{1}{2}$ representation. In recent years, several groups discussed similar problems [3, 4, 5] for the Maxwell and Dirac equations. They showed that several nonrelativistic limits may exist in these cases.

In the present paper, we focus on the nonrelativistic limit of covariant nonlinear self-consistent effective field theories, i.e. those based on density functionals. These cases are to be distinguished from other effective field theories which rely on a straighforward expansion (see e.g. [6]) and for which the following consideration do not apply. Realizations of self-consistent field theories include e.g. the model by Duerr [7], Heisenberg's nonlinear spinor theory [8], the Nambu-Jona-Lasinio (NJL) model [9], and effective ϕ^4 theories [10]. In nuclear physics, the relativistic mean-field (RMF) model [11, 12] and the Skyrme-Hartree-Fock (SHF) approach [12] are typical examples.

In this manuscript, we will discuss the link between such a non-linear relativistic theory and its nonrelativistic counterpart. It is found that a straightforward nonrelativistic reduction of a covariant ansatz up to $(v/c)^2$ yields a result which violates Galileian invariance. We will develop and justify a non-relativistic mapping going up to $(v/c)^2$ that leads to the correct Galilean-invariant counterpart.

II. NONRELATIVISTIC REDUCTION

A. The goal

A relativistic effective field theory expresses the configuration in terms of Dirac four-spinor wavefunctions ψ_α for each state α. For the further developments, it is useful to express it explicitly through upper and lower two-spinor components as

$$\psi_\alpha = \begin{pmatrix} \varphi_\alpha^{(\mathrm{u})} \\ \varphi_\alpha^{(\mathrm{d})} \end{pmatrix}. \tag{1}$$

For means of simplicity and lucidity we choose a simple covariant self-consistent effective field theory involving point couplings between its degrees of freedom, i.e., the

4-component spinors. It reads

$$\mathcal{L}_{\mathrm{c}} \;=\; \mathcal{L}_{\mathrm{free}} + \frac{c_s}{2}\varrho_s^2 + \frac{c_s}{2}\varrho_\mu\varrho^\mu, \tag{2}$$

with

$$\varrho_s \;=\; \sum_\alpha \bar{\psi}_\alpha\psi_\alpha = \sum_\alpha \left[\varphi_\alpha^{(\mathrm{u})\dagger}\varphi_\alpha^{(\mathrm{u})} - \varphi_\alpha^{(\mathrm{d})\dagger}\varphi_\alpha^{(\mathrm{d})}\right] \;, \tag{3a}$$

$$\varrho_0 \;=\; \sum_\alpha \bar{\psi}_\alpha\gamma_0\psi_\alpha = \sum_\alpha \left[\varphi_\alpha^{(\mathrm{u})\dagger}\varphi_\alpha^{(\mathrm{u})} + \varphi_\alpha^{(\mathrm{d})\dagger}\varphi_\alpha^{(\mathrm{d})}\right], \tag{3b}$$

$$\boldsymbol{\varrho} \;=\; \sum_\alpha \bar{\psi}_\alpha\boldsymbol{\gamma}\psi_\alpha = \sum_\alpha \left[\varphi_\alpha^{(\mathrm{u})\dagger}\boldsymbol{\sigma}\varphi_\alpha^{(\mathrm{d})} + \varphi_\alpha^{(\mathrm{d})\dagger}\boldsymbol{\sigma}\varphi_\alpha^{(\mathrm{u})}\right] \tag{3c}$$

with $\gamma_\mu = (\gamma_0, \boldsymbol{\gamma})$ the four-vector of Dirac matrices [13]. The relativistic functional appears simple because the kinetic and spin-orbit terms are implicit in the scalar and vector densities, as we shall see. For simplicity of notation, we will drop the index label α in the following, identifying, e.g., the scalar density with $\varrho_s = \bar{\psi}\psi$ and similarly for all other densities and currents.

A key point in such approaches is, of course, correct normalization of the densities and consequently the wavefunctions. This implies

$$\int \mathrm{d}^3r\bar{\psi}\gamma_0\psi = 1 \tag{4}$$

to guarantee invariance under Lorentz transformations. This may be surprising, since it involves the zeroth component of a four vector, but is inevitable to counter the relativistic contraction of the purely spatial volume element d^3r [13]. We will see that this is the key problem with straightforward expansions and at the same time the key to the solution.

The goal is now to obtain a nonrelativistic functional based on the densities (6) from the relativistic parent functional (2) in a nonrelativistic limit. Solutions in the positive-energy branch (particle-like) are distinguished by a dominance of $\varphi^{(\mathrm{u})}$ over $\varphi^{(\mathrm{d})}$. The strategy is thus to eliminate $\varphi^{(\mathrm{d})}$ and identify the upper component with the classical two-spinor wavefunction, $\varphi^{(\mathrm{u})} \longleftrightarrow \varphi^{(\mathrm{cl})}$, which should obey the normalization condition

$$\int \mathrm{d}^3r\big|\varphi^{(\mathrm{cl})}\big|^2 = 1. \tag{5}$$

We can thus express ψ in terms of $\varphi^{(\mathrm{cl})}$. Inserting that into the coupling Lagrangian density (2) should produce the desired limit.

In the course of obtaining the nonrelativistic limit we expect to formulate the nonrelativistic counterpart in terms of the following densities and currents:

$$\rho \;=\; \sum_\alpha \big|\varphi_\alpha^{\mathrm{cl}}\big|^2, \quad \tau \;=\; \sum_\alpha \big|\nabla\varphi_\alpha^{\mathrm{cl}}\big|^2, \tag{6a}$$

$$\mathbf{J} \;=\; -\frac{i}{2}\sum_\alpha [\varphi_\alpha^{\mathrm{cl}\dagger}(\nabla\times\sigma)\varphi^{\mathrm{cl}} - (\nabla\times\sigma\varphi_\alpha^{\mathrm{cl}})^\dagger\varphi_\alpha^{\mathrm{cl}}], \tag{6b}$$

$$\mathbf{j} \;=\; -\frac{i}{2}\sum_\alpha [\varphi_\alpha^{\mathrm{cl}\dagger}\nabla\varphi_\alpha^{\mathrm{cl}} - (\nabla\varphi_\alpha^{\mathrm{cl}})^\dagger\varphi_\alpha^{\mathrm{cl}}], \tag{6c}$$

$$\boldsymbol{\sigma} \;=\; \sum_\alpha \varphi_\alpha^{\mathrm{cl}\dagger}\sigma\varphi_\alpha^{\mathrm{cl}}, \tag{6d}$$

where ρ, τ and $\mathbf{J}$ are time-even while $\mathbf{j}$ and $\boldsymbol{\sigma}$ are time-odd. Note that time-even and time-odd terms appear in particular combinations, a feature which is crucial to render the functional invariant under Galilean transformations [14].

B. Problems with v/c expansions

Nonrelativistic limits are usually obtained by straightforward expansion in orders v/c or p/m, respectively, e.g. in the Foldy-Wouthuysen transformation ([13], for the nuclear case see [11, 15]). We briefly review the steps from [11]. One starts from the relativistic equations of motion, for the present model derived from the Lagrangian (2)

$$0 \;=\; (\mathrm{i}\gamma^\mu\partial_\mu - m + S + \gamma^\mu V_\mu)\,\psi_\alpha, \tag{7a}$$

$$S \;=\; -c_s\varrho_s, \quad V_\mu = c_v\varrho_\mu. \tag{7b}$$

with the self-consistent scalar and vector potentials S and V_μ, the latter decomposing as

$$V_\mu = (V_0, -\mathbf{V}) = (c_v\varrho_0, -c_v\boldsymbol{\varrho}). \tag{8}$$

We insert the decomposition (1) and solve the lower-component equation for $\varphi^{(\mathrm{d})}$. Keeping only terms up to order p/m yields

$$\varphi^{(\mathrm{d})} \;=\; B_0\sigma\cdot(\hat{\mathbf{p}} - \mathbf{V})\varphi^{(\mathrm{u})}, \tag{9a}$$

$$B_0 \;=\; \frac{1}{2m + S - V_0} \approx \frac{1}{2m}, \tag{9b}$$

where $\hat{\mathbf{p}} = -\mathrm{i}\nabla$. The approximation $B_0 = 1/m$ ignores the density dependence in B_0. It suffices for the present studies. The form (9) violates the normalization (4) to second order in p/m. The procedure of [11] restores (ortho-)normalization at operator level by first imposing the relativistic normalization (4) up to $(p/m)^2$

$$1 \;=\; \int \mathrm{d}^3r\,\varphi^{(\mathrm{u})\dagger}\left[1 + \hat{\mathcal{T}}\right]\varphi^{(\mathrm{u})}$$

$$\hat{\mathcal{T}} \;=\; B_0^2\left(\sigma\cdot(\mathbf{p} - \mathbf{V})\right)^2 . \tag{10}$$

We introduce $\varphi^{(\mathrm{cl})}$ so as to recover the nonrelativistic normalization (5). Thus we identify

$$1 \;=\; \int \mathrm{d}^3r\,\underbrace{\varphi^{(\mathrm{u})\dagger}\left[1 + \hat{\mathcal{T}}\right]^{1/2}}_{\varphi^{((\mathrm{cl})\dagger}}\underbrace{\left[1 + \hat{\mathcal{T}}\right]^{1/2}\varphi^{(\mathrm{u})}}_{\varphi^{((\mathrm{cl})}}$$

We expand in second order of p/m and obtain

$$\varphi^{(\mathrm{u})} = \Big[1 - \frac{1}{2}\hat{\mathcal{T}}\Big]\varphi^{(\mathrm{cl})} \tag{11}$$

which, together with relation (9a), provides a complete description of ψ in terms of $\varphi^{(\mathrm{cl})}$ up to order p/m. Using the relation $(\hat{\boldsymbol{\sigma}}\cdot\mathbf{A})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{B}) = \mathbf{A}\cdot\mathbf{B} - \mathrm{i}\mathbf{A}\cdot(\hat{\boldsymbol{\sigma}}\times\mathbf{B})$ yields

$$\frac{\hat{\mathcal{T}}}{2B_0^2} = (\hat{\mathbf{p}} - \mathbf{V})^2 + \mathrm{i}(\hat{\mathbf{p}} - \mathbf{V})\cdot(\hat{\boldsymbol{\sigma}}\times(\hat{\mathbf{p}} - \mathbf{V}))\,. \tag{12}$$

We insert eqs. (8), (11) and (12) into expressions (3) for the relativistic densities and finally obtain up to order $(p/m)^2$

$$\varrho_s = \rho - 2B_0^2\Big[\tau - \nabla\cdot\mathbf{J} - c_v\boldsymbol{\varrho}\cdot(2\mathbf{j}+\nabla\times\boldsymbol{\sigma}) + c_v^2\boldsymbol{\varrho}^2\rho\Big] \tag{13a}$$

$$\varrho_0 = \rho, \quad \boldsymbol{\varrho} = 2B_0\Big(\mathbf{j} - c_v\boldsymbol{\varrho}\rho + \frac{1}{2}\nabla\times\boldsymbol{\sigma}\Big). \tag{13b}$$

The same relations are obtained when going through the Foldy-Wouthuysen transformation up to second order. Note that the spatial part of the vector density $\boldsymbol{\varrho}$ is at least of first order such that the correction from $\hat{\mathcal{T}}$ would be of third order and is discarded. The result is correct for stationary states where all time-odd densities and fields vanish, i.e. where $\mathbf{j} = 0$, $\boldsymbol{\sigma} = 0$, and $\mathbf{V} = 0$. The expressions (13) when inserted into the Lagrangian (2) produce a serious defect: the emerging nonrelativistic Lagrangian is not Galilei invariant. The Lorentz invariant scalar density ρ_s does not translate to a Galileian invariant expression and the same happens for the combination $\varrho_0^2 - \boldsymbol{\varrho}^2$. Correct expressions should form the Galileian invariant combinations $\rho\tau - \mathbf{j}^2$ and $\rho\nabla\cdot\mathbf{J} + \mathbf{j}\cdot(\nabla\times\boldsymbol{\sigma})$ as we will see later.

In order to elucidate the problem, we consider the transformation properties for the simple case of an explicit boost of the whole system. Let us start with a well-checked situation, a stationary state for which $\mathbf{j} = 0$ and $\boldsymbol{\sigma} = 0$. We boost the system by a constant velocity $\mathbf{u}$ (in units of c). All quantities in the boosted system will be distinguished by a prime. The normalization (4) becomes in the boosted frame

$$1 = \int \mathrm{d}^3r'\,\varrho_0' = \int \mathrm{d}^3r\sqrt{1-\mathbf{u}^2}\,\frac{\varrho_0}{\sqrt{1-\mathbf{u}^2}}. \tag{14}$$

Note the transformation of the volume element exactly countering that of the density ϱ_0. The volume dilatation factor is negligible to order u^1 but contributes in second order. The above nonrelativistic expansion to second order had violating terms at that order. The example shows that the mistake lies in neglecting a second order correction of the volume element in the normalization condition.

C. Map instead of expansion

The previous discussion shows that a straightforward nonrelativistic expansion with all kinetic contributions is consistent only up to first order p/m (or boost velocity u, respectively), while the crucial relativistic corrections to a classical Schrödinger equation emerge from second order terms, namely spin-orbit coupling and effective-mass terms. These require a special handling of the normalization condition like in the example of Eq. (14). We thus leave the straightforward paths of p/m expansion to now aim at a generalized mapping of the relativistic functional into a nonrelativistic one, trying to incorporate all second-order effects.

The key point is accounting for the relativistic volume-element compression as in Eq. (14). To deal with arbitrary nonstationary situations we need to allow a boost velocity field. Thus we modify the normalization condition (4) to display explicitly the compression factor with respect to the local boost velocity, which in turn is expressed in terms of the classical densities and currents:

$$1 = \int \mathrm{d}^3r\sqrt{1-\mathbf{u}^2(\mathbf{r})}\,\varrho_0 \tag{15a}$$

$$\mathbf{u}(\mathbf{r}) = \frac{\boldsymbol{\varrho}}{\varrho_0} \equiv \frac{2B_0}{\rho}\Big(\mathbf{j} - \mathbf{V}\rho + \frac{1}{2}\nabla\times\boldsymbol{\sigma}\Big). \tag{15b}$$

The expansion (11) thus is slightly modified to the map

$$\begin{aligned}\varphi^{(\mathrm{u})} &= \Big[1 - \frac{1}{2}\hat{\mathcal{T}}\Big]\varphi^{(\mathrm{cl})}(1 + \frac{1}{4}\mathbf{u}^2) \\ &\approx \Big[1 - \frac{1}{2}\hat{\mathcal{T}} + \frac{1}{4}\mathbf{u}^2\Big]\varphi^{(\mathrm{cl})}.\end{aligned} \tag{15c}$$

Things now proceed as in section II B, but the term $\propto \mathbf{u}^2$ cancels the unwanted one in ϱ_s and adds a desired one in ϱ_0. This now leads to the consistent result

$$\begin{aligned}\varrho_s &= \rho - \frac{2B_0^2}{\rho}\Big[\rho\tau - \mathbf{j}^2 - \rho\nabla\cdot\mathbf{J} - \mathbf{j}\cdot(\nabla\times\boldsymbol{\sigma}) \\ &\qquad - \frac{1}{4}(\nabla\times\boldsymbol{\sigma})^2\Big]\end{aligned} \tag{16a}$$

$$\varrho_0 = \rho + \frac{2B_0^2}{\rho}\Big(\mathbf{j} - \mathbf{V}\rho + \frac{1}{2}\nabla\times\boldsymbol{\sigma}\Big)^2, \tag{16b}$$

$$\boldsymbol{\varrho} = 2B_0\Big(\mathbf{j} - \mathbf{V}\rho + \frac{1}{2}\nabla\times\boldsymbol{\sigma}\Big). \tag{16c}$$

The scalar density shows the wanted Galilean-invariant combinations and the vector density reproduces the correct invariance property, $\varrho_\mu\varrho^\mu = \rho_0^2 - \boldsymbol{\varrho}^2 \approx \rho^2$, up to terms of second order, of course. We thus insert the mapped scalar and vector densities (16) into the interaction Lagrangian density (2), getting (up to second order)

$$\begin{aligned}\mathcal{L}_\mathrm{c} &= \frac{c_s + c_v}{2}\rho^2 - 2c_sB_0^2\Big[\rho\tau - \mathbf{j}^2 \\ &\qquad -(\rho\nabla\cdot\mathbf{J} + \mathbf{j}\cdot(\nabla\times\boldsymbol{\sigma})) - \frac{1}{4}(\nabla\times\boldsymbol{\sigma})^2\Big].\end{aligned} \tag{17}$$

That result is manifestly Galilean invariant.

It has the same form as the basic version of the Skyrme Hamiltonian density [12] that is being employed for the

description of finite nuclei. This comes as no surprise, since the covariant Lagrangian that we started with, Eq. (2), consists of the basic terms of the Lagrangian of the point-coupling variant of the RMF model for nuclear structure, RMF-PC [16]. Both SHF and RMF-PC are formulated in terms of point couplings of spinors and thus display this close relationship. This relation between covariant models and their Galilean-invariant counterparts is of importance when one analyzes, for example, spin excitation mechanisms in nonrelativistic time-dependent Hartree Fock employing the Skyrme functional [17].

Compared with the Skyrme functional, however, there is one additional term $\propto (\nabla\times\boldsymbol{\sigma})^2$ which is an allowed term in the Skyrme functional, but usually neglected. There is, however, no gradient term $\propto \rho\Delta\rho$ which is mandatory for the description of finite-size systems. That is no surprise because we had started from a simplified Lagrangian without gradient terms. The more complete model would also include terms such as $\rho_s\Delta\rho_s$ and $\rho_\mu\Delta\rho^\mu$, whose expansion proceeds quite similarly and in the nonrelativistic limit yields gradient terms. There is a subtle difference, though: the nonrelativistic mapping would also produce gradient kinetic terms like $\rho\Delta\tau$, which are neglected assuming that the gradient correction as such is small and second order relativistic corrections to it are negligible. The counter argument is that there are two quite different notions of smallness involved here that may not be combined.

III. CONCLUSION

We have studied the nonrelativistic limit of a self-consistent relativistic theory with the aim of recovering a minimum of relativistic effects, the spin-orbit force, together with Galilean invariance in the resulting nonrelativistic theory. Note that Galilean invariance requires keeping all time-odd terms, which play a crucial role in the formulation of dynamics. This applies to the spatial components of the relativistic vector density as well as to the current and spin densities in the nonrelativistic domain. Previous derivations rarely studied the full dynamical case.

The procedure started out with a straightforward v/c expansion, which encountered inconsistencies, because the spin-orbit term appears only in second order of v/c while Galilean invariance comes out correctly only to first order. The key finding is that a strictly nonrelativistic theory is not easily compatible with the appearance of a spin-orbit term. To be more precise: *for relativistic effective field theories, based on density functionals, it is not possible to derive a sufficiently complete nonrelativistic theory by mere expansion and order counting.* Instead the more general concept of a nonrelativistic mapping is needed, namely to derive an effective nonrelativistic theory which includes as many features of the given relativistic theory as desired. Starting from the simple consideration of Lorentz contraction of the spatial volume element, we have derived such a mapping for a covariant self-consistent model. This mapping manages to provide a manifestly Galilean invariant theory which correctly incorporates the spin-orbit and effective-mass terms and, if one starts with the RMF-PC model in nuclear physics, merges into the widely used Skyrme-Hartree-Fock approach when neglecting the involved density dependences of the spin-orbit and effective-mass term. Extensions of the scheme developed here are in progress.

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[1] I.L. Foldy and S.A. Wouthuysen, Phys. Rev. **78**, 29 (1950).
[2] R. Gilmore, *Lie Groups, Lie Algebras and some of their applications*, (Wiley-Interscience Publication, New York, 1974).
[3] M. de Montigny, J. Niederle and A. G. Nikitin, math-ph/0604002
[4] H. R. Brown, P. Holland, Am. J. Phys. **67** (1999), 204.
[5] P. Holland and H. R. Brown, Studies in History and Philosophy of Modern Physics **34**A (2003), 161.
[6] S. Weinberg, Physica A **96** (1979), 327
[7] H.-P. Duerr, Phys. Rev. **103** (1956), 469
[8] W. Heisenberg, *Introduction to the Unified Field Theory of Elementary Particles*, (Wiley, N.Y., 1967).
[9] S. Klevansky, Rev. Mod. Phys. **64** (1992), 649
[10] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, (Carlderon Press, Oxford, 2002).
[11] P. G. Reinhard, Rep. Prog. Phys. **52** (1989), 439.
[12] M. Bender, P.-H. Heenen, P.-G. Reinhard, Rev. Mod. Phys. **75** (2003), 121.
[13] C. Itzykson and J. B. Zuber, Quantum Field Theory **Mc Graw-Hill**, pg:69 (1980).
[14] J. Y. Engel, D. M. Brink, K. Goeke, S. J. Krieger and D. Vautherin, Nucl. Phys. A **249** (1975), 215.
[15] M. Thies, Nucl. Phys. A **460** (1986), 636.
[16] T. Buervenich, D. G. Madland, J. A. Maruhn and P.-G. Reinhard Phys. Rev. C **65**, 044308 (2002)
[17] J. A. Maruhn, P.–G. Reinhard, P. D. Stevenson, and M. R. Strayer, Phys.Rev. **C74** (2006), 027601